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The Rotation of LAGEOS

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In view of the need of an accurate modelling of nongravitational forces on laser-tracked satellites, it is important to understand their rotational dynamics, which determines the temperature anisotropy and the ensuing radiation recoil effects. We propose a model of the torques acting on LAGEOS due to eddy currents and gravity gradient. The electromotive forces induced in the spacecraft by its rotation in the magnetic field of the Earth dissipate angular momentum and produce a precession of the spin axis; the oblate spacecraft will precess in the gravitational field of the Earth at a rate proportional to the rotation period. Therefore the gravitational torques become more and more important with time and eventually may produce a chaotic dynamics. The predicted evolution of the spin period agrees very well with the few experimental data available and corresponds to an approximately exponential growth rate of about 3 years.

1. Introduction

In the 1960s, space physicists devoted a good deal of attention to the natural torques acting on spacecraft, with the hope of controlling their attitude without active propulsion systems; for example, by activating magnets inside a spacecraft orbiting in the magnetic field of the Earth one could control its attitude [*Wilson*, 1959, 1960, 1964; *Hecht and Manger*, 1964]. Colombo investigated the rotation of Explorer XI and studied the torques induced by the gravitational gradient [*Colombo*, 1964] and by permanent magnetization with hysteresis effects [*Colombo*, 1967]. *Colombo* [1967] also give a short and semiempirical discussion on the loss of angular momentum due to eddy currents, one of the main topics of the present work.

With the advent of active and precise attitude control systems, the problem of natural torques on spacecraft lost its main practical interest and laid neglected for some time, but now there is a strong, new motivation from the need for a full understanding of the very weak forces acting on laser-tracked satellites like LAGEOS [*Cohen et al.*, 1985]. It is now possible to measure its long-term acceleration to an accuracy of $\approx 10^{-10}$ cm/s^2; indeed, this very high accuracy, and a good theoretical model to match it, is required by the exceedingly precise range measurements available (down to 1 cm) and the need for a good orbital solution valid for a long time. A very accurate knowledge of the orbital elements of LAGEOS is important for geodetic applications. Indeed, at present, the anomalous, along-track acceleration, which is the main cause for the uncertainty in the position of LAGEOS, is evaluated with a least squares fit which uses as unknown parameters its averages over an interval from 1 to 2 weeks. However, this time interval will become shorter as the accuracy in the distance measurements increases, leading to an unpleasant proliferation of unknown parameters; in fact, a new type of data analysis in which each pass is independently fitted has been proposed [*Milani and Melchioni*, 1989]. A better understanding of the nongravitational forces will become more and more important in the future.

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Recently, *Ciufolini* [1986] proposed using two LAGEOS satellites with supplementary inclinations to measure the gravitomagnetic force due to the angular momentum of the Earth; the final error of this relativistic experiment is limited by our "poor" knowledge of the nongravitational forces acting on the two spacecraft [*Ciufolini*, 1987].

At this very high accuracy, several complex and little known effects come into play: one could say, the nongravitational forces acting on LAGEOS are an interesting problem of space physics in itself [*Rubincam*, 1982, 1987; *Rubincam et al.*, 1987; *Afonso et al.*, 1980, 1985, 1990; *Anselmo et al.*, 1983; *Barlier et al.*, 1986; *Farinella et al.*, 1990; *Ciufolini et al.*, 1990]. We have a drag due to the interaction with the charged particles; the radiation pressure from the Sun, as modified by the eclipses; the radiation pressure from the Earth, both in the optical and the infrared band; and, finally and crucially, the reaction due to the anisotropic radiation emitted by the spacecraft with an inhomogeneous temperature distribution. In particular, *Rubincam* [1987] and *Rubincam et al.* [1987] has shown that the observed, long-term change in the semimajor axis of LAGEOS could be explained by the "radiation rocket" effect due to the terrestrial infrared radiation on a spinning spacecraft, although the direct Earth albedo radiation pressure could also give a similar effect [*Anselmo et al.*, 1983]. Since the temperature anisotropy is determined by the vectorial angular velocity, a theoretical model for its time evolution is required.

The "radiation rocket" effect may play a role for a laser-tracked satellite on three counts: (1) the solar heating, especially as a consequence of eclipses, which produce long-term effects on the orbital elements; (2) the infrared radiation of the Earth [*Rubincam*, 1987]; and (3) for slowly rotating satellites, the "daily" Yarkowsky effect due to a lack of temperature uniformity along their equator, producing an additional force orthogonal to the spin axis. The knowledge of the spin vector of the spacecraft is essential for an understanding of these effects.

For a passive, nonferromagnetic spacecraft like LA-

GEOS, the main acting torques are (1) the parasitic currents generated in its conducting body by the $\mathbf{v} \times \mathbf{B}$ force due to the rotational velocity and (2) the gradient of the gravitational field of the Earth acting upon an oblate mass distribution. Surprisingly enough, the general problem of the torque acting upon a conductor which spins in an external magnetic field does not seem to have been discussed in the literature. Exact solutions have been found, however, for a sphere and spherical shell rotating in a uniform magnetic field [*Halverson and Cohen*, 1964, and references therein; *Landau and Lifshitz*, 1960].

In our model the angular velocity $\boldsymbol{\omega}(t)$ is governed by a first-order, nonlinear vectorial differential equation characterized by three time scales: the magnetic decay time ν_m^{-1}, the nodal period $P_N = 2\pi/|\dot{\Omega}|$, and the "Hipparcos" precession period $2\pi/\omega_p$ due to the gravitational gradient. For LAGEOS, the first two constant are of the same order of magnitude; the last quantity is initially much larger, but eventually, it becomes comparable. We attack the problem with a combination of analytical approach (multiple time scales [*Bogoliubov and Mitropolski*, 1961]) and numerical integration and show the great variety and complexity of results in different regions of parameter space.

Some measurements of the rotation period of LAGEOS are available, using both a coherent laser and microwave radar. One laser measurement is known [*Sullivan*, 1980] and gave a rotation period of 1.44 s on April 11, 1979 (approximately 3 years after launch). At launch on May 4, 1976, the period T was nominally 0.6 s [*Ordahl*, 1975] and the axis lay, again nominally, in the orbital plane at an angle $\theta_0 \approx 22°$ with the polar axis ($\cos\theta_0 = \pm 0.92675$) [*Rubincam*, 1987]. The period seems to increase approximately exponentially with a time constant of about 3 y (M. Gaposchkin, private communication, 1989); in 1989 it should be about

$$0.6 \exp(13/3) = 46 \text{ s}$$

2. MAGNETIC TORQUE ON LAGEOS

In the rotating frame the Earth's magnetic field changes with the rotation frequency ω and induces in the body of the satellite eddy currents up to a depth of about

$$\delta = \frac{c}{(2\pi\omega\sigma)^{1/2}} \tag{1}$$

where σ is the electrical conductivity. Although the complex structure of LAGEOS requires a delicate discussion to assess the importance for the torque of the penetration depth (1) (see later), using in equation (1) the value $\sigma = 2.2 \times 10^{17}$ s^{-1} [*Harper*, 1970] for aluminum alloy UNI 6061T6 (the material of the outer shell), we get

$$\delta = 7.9 \text{ cm } (T/0.6 \text{ s})^{1/2} \tag{2}$$

which remains smaller than the radius $\rho = 30$ cm for a long time after launch. Notwithstanding, as we shall discuss later, we have good reasons to believe that the effect upon the torque of a finite penetration depth is negligible. In general, the magnetic torque

$$\boldsymbol{\Gamma}_m = \mathbf{m} \times \mathbf{B} \tag{3}$$

is determined by the magnetic moment $\mathbf{m}$. For an axially symmetric conductor rotating with angular velocity $\boldsymbol{\omega}$ around its axis of symmetry e in an external magnetic field $\mathbf{B}$, $\mathbf{m}$, linear in $\mathbf{B}$, must be a linear combination of $\hat{\boldsymbol{\omega}} \times \mathbf{B}$ and $\hat{\boldsymbol{\omega}}\,(\mathbf{B} \cdot \hat{\boldsymbol{\omega}})$:

$$\mathbf{m} = V\alpha'\hat{\boldsymbol{\omega}}(\hat{\boldsymbol{\omega}} \cdot \mathbf{B}) - V\alpha''\mathbf{B} \times \hat{\boldsymbol{\omega}} \tag{4}$$

$\alpha' + i\alpha'' = \alpha$ is the average (complex and dimensionless) polarizability per unit volume and V is the volume of the conductor. Hence the magnetic torque is quadratic in, and orthogonal to, $\mathbf{B}$:

$$\boldsymbol{\Gamma}_m = V\alpha'\hat{\boldsymbol{\omega}} \times \mathbf{B}(\hat{\boldsymbol{\omega}} \cdot \mathbf{B}) - V\alpha''(\mathbf{B} \times \hat{\boldsymbol{\omega}}) \times \mathbf{B} = \boldsymbol{\Gamma}' + \boldsymbol{\Gamma}'' \tag{3'}$$

We neglect here any deviation of the rotation axis from the symmetry axis e. The assumption $e \parallel \boldsymbol{\omega}$ is justified by the fact that a rotation about the principal axis corresponding to the maximum momentum of inertia is stable. For LAGEOS, in fact, the angular velocity was initially oriented along the symmetry axis, which is also a stable principal axis. *Landau and Lifshitz* [1960] have determined the functions $\alpha'(\rho/\delta)$ and $\alpha''(\rho/\delta)$ for a uniform sphere of radius ρ; in the low frequency limit ($\rho \ll \delta$) they obtain

$$\alpha' = -\frac{4\pi}{105}\frac{\rho^4\sigma^2\omega^2}{c^4} = -\frac{1}{105\pi}\left(\frac{\rho}{\delta}\right)^4 \tag{5}$$

$$\alpha'' = \frac{\rho^2\sigma\omega}{10c^2} = \frac{1}{20\pi}\left(\frac{\rho}{\delta}\right)^2 \tag{6}$$

In this limit, the two contribution to the torque are proportional, respectively, to

$$V\frac{\rho^4}{\delta^4}B^2 \propto \frac{V\rho^4B^2\sigma^2\omega^2}{c^4} \qquad V\frac{\rho^2}{\delta^2}B^2 \propto \frac{V\rho^2B^2\sigma\omega}{c^2} \tag{7}$$

and the second dominates. For LAGEOS, we set, in general,

$$\alpha' = -\frac{1}{105\pi}\left(\frac{\rho}{\delta}\right)^4 \beta'\left[\left(\frac{\rho}{\delta}\right)^2\right] \tag{8}$$

$$\alpha'' = \frac{1}{20\pi}\left(\frac{\rho}{\delta}\right)^2 \beta''\left[\left(\frac{\rho}{\delta}\right)^2\right] \tag{9}$$

The two dimensionless functions β' and β'' determine by how much the electrical properties of the satellite differ from those of a homogeneous sphere of the same radius, at low frequency. The argument of the functions β' and β'' is $x = \rho^2/\delta^2$, and not ρ/δ, as it is for a sphere. In the low-frequency limit we are primarily interested in the parameter $\beta''(0)$ and, secondarily, in the parameters $\beta'(0)$ and $(d\beta''/dx)_{x=0}$, which produce contribution to the torque formally of the same order. An important part of our work is to determine their values for LAGEOS from the measurements of the period.

Equations (4) and (8) and (9) show that in the low-frequency limit, the main contribution to the torque comes from the imaginary part α'' of the complex polarizability. The other term, being orthogonal to $\boldsymbol{\omega}$, does not produce work; its effect, reduced by the small value of the numerical coefficient, is a precession with frequency:

$$\boldsymbol{\Omega}_m = -\frac{V\alpha'}{C\omega}(\mathbf{B} \cdot \hat{\boldsymbol{\omega}})\mathbf{B} \tag{10}$$

where C is the moment of inertia with respect to the symmetry axis e. This term changes the period of the satellite only indirectly, through the change in the direction of rotation.

Whether or not the magnetic torque acting on LAGEOS falls in the low-frequency regime cannot be said a priori. Were LAGEOS a homogeneous aluminum sphere, because of the large initial ratio ρ/δ, the functions β' and β'' would be required in a finite interval; however, there are at least three arguments which support the applicability of the low-frequency limit. First of all, LAGEOS has a complex internal structure. It is made by two aluminum hemispherical shells embedding an internal brass core. Although these three parts are mounted in close mechanical contact, the electrical conductivity at the inner boundaries is likely to be very poor, due to the oxide layer on the aluminum surfaces. In other words, the three main parts of the satellite may, to a good degree, be considered as three distinct conductors with no (or poor) electrical contact. The effective ratio ρ/δ is hence smaller than the one computed with the bare geometrical radius of the satellite. Second, the main contribution to the magnetic torque comes from currents flowing in the external layers of the body, where the "skin effect" is less important. Third, in the case of the homogeneous sphere the values of α' and α'' differ, for $\rho/\delta < 1.5$ by less than 20% from the values computed using the asymptotic expansions (5) and (6). Hence we can expect that the low-frequency limit is a good approximation even at the beginning of the mission. In the following, we concentrate on this limit; as it will be shown in section 4, equation (51), the excellent agreement of the theory with the observations confirms, a posteriori, the validity of this choice.

In this limit, the induced magnetic moment of the body is simply

$$\mathbf{m} = -V\frac{\sigma\rho^2}{20\pi c^2}\beta''(0)\mathbf{B}\times\boldsymbol{\omega} \tag{11}$$

and the torque

$$\boldsymbol{\Gamma}'' = \mathbf{m}\times\mathbf{B} = -V\frac{\sigma\rho^2}{20\pi c^2}\beta''(0)\boldsymbol{\omega}\cdot(\mathbf{BB}-\mathbf{1}B^2) \tag{12}$$

The parameter $V\sigma\rho^2\beta''(0)$ is to be understood as an effective value, to be determined from the observations. For definiteness, we set ρ = 30 cm and $\sigma = 2.5\times10^{27}$ s^{-1} and leave $\beta''(0)$ free.

3. Magnetic Slowing Down of LAGEOS

The magnetic field experienced by LAGEOS along its orbit is not constant. Since the angular velocity changes very little in an orbital period, we can average the torque (12) for the dipole magnetic field

$$\mathbf{B} = \nabla\frac{\mathbf{d}\cdot\mathbf{r}}{r^3} = \frac{r^2\mathbf{d} - 3\mathbf{r}(\mathbf{r}\cdot\mathbf{d})}{r^5} \tag{13}$$

At the altitude of LAGEOS this is a good approximation. The magnetic dipole vector $\mathbf{d}$ makes an angle D of about 10° with the axis of the Earth $\mathbf{E}$ and rotates around it with the period of a day. Although the finite value of the angle D can be easily taken into account (see Appendix A), we shall, for the sake of simplicity, neglect it here, thereby making an error of order $D^2 \approx 0.037$. When the orbit is circular, the averaging operation can be done on the basis of symmetry principles alone, without any explicit integration (see Appendix B); the required positive definite matrix

$$\langle B^2\delta_{ij} - B_iB_j\rangle = \frac{d^2}{a^6}\beta_{ij} \tag{14}$$

can be expressed in terms of the magnetic dipole d, the orbital radius a, and the normal $\mathbf{n}$ to the orbital plane. The orbital inclination I is defined by

$$\cos I = \mathbf{n}\cdot\mathbf{E} \tag{15}$$

In Appendix B we show that

$$\beta_{ij} = \tfrac{1}{8}(11 - 3\cos^2 I)\delta_{ij} - \tfrac{1}{4}E_iE_j + \tfrac{9}{8}(1 - 3\cos^2 I)n_in_j + \tfrac{3}{4}(E_in_j + E_jn_i)\cos I \tag{16}$$

The vector $\mathbf{n}$ precesses around the polar vector $\mathbf{E}$ with the nodal precision, at the rate $\dot{\Omega}$ (for LAGEOS, 0.343°/d). In the frame where the node is at rest, we have

$$\mathbf{E} = \begin{pmatrix}0\\0\\1\end{pmatrix} \qquad \mathbf{n} = \begin{pmatrix}\sin I\\0\\\cos I\end{pmatrix} \tag{17}$$

and the matrix β_{ij} has the form

$$\beta_{ij} = \begin{pmatrix}b_1 & 0 & b_2\\0 & b_3 & 0\\b_2 & 0 & b_4\end{pmatrix} \tag{18}$$

where

$$\begin{aligned}b_1 &= \tfrac{1}{8}(20 - 39\cos^2 I + 27\cos^4 I)\\ b_2 &= \tfrac{3}{8}(5 - 9\cos^2 I)\cos I\sin I\\ b_3 &= \tfrac{1}{8}(11 - 3\cos^2 I)\\ b_4 &= \tfrac{3}{8}(3 - 4\cos^2 I - 9\cos^4 I)\end{aligned} \tag{19}$$

Neglecting, for the time being, the gravitational torque, the mean rotational vector ω_i fulfils

$$\frac{1}{\nu_m}\frac{d\omega_i}{dt} = -\beta_{ij}\omega_j \tag{20}$$

where

$$\nu_m = V\frac{\sigma\rho^2\beta''(0)}{20\pi Cc^2}\frac{a^2}{d^6} \tag{21}$$

is the value, in the low-frequency limit, of the magnetic decay frequency. The eigenvalues $\beta_{(i)}$ of the matrix β_{ij} and their eigenvectors $\mathbf{u}_{(i)}$ play a crucial role. $\mathbf{u}_{(1)}$ and $\mathbf{u}_{(3)}$ lie in the plane of $\mathbf{n}$ and $\mathbf{E}$; their eigenvalues are the roots of

$$(b_1 - \beta)(b_4 - \beta) - b_2^2 = 0 \tag{22}$$

The last eigenvector $\mathbf{u}_2$, corresponding to the eigenvalue $\beta_2 = b_3$, is orthogonal to the plane $(\mathbf{n}, \mathbf{E})$

$$\mathbf{u}_{(2)} = \mathbf{N} = \frac{\mathbf{E}\times\mathbf{n}}{|\sin I|} \tag{23}$$

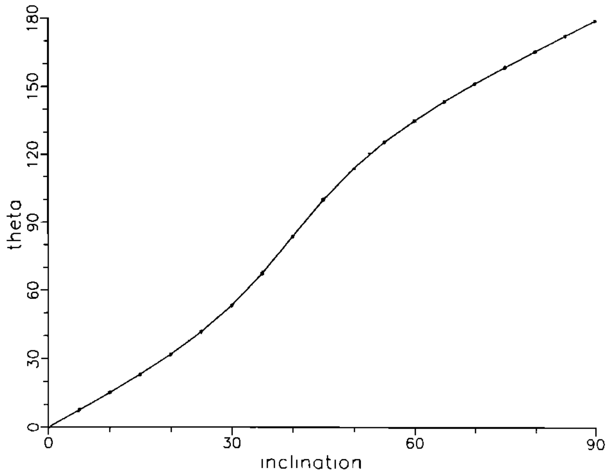

Fig. 1. Angle between the Earth's rotation axis and the eigenvector of the matrix β_{ij} corresponding to the least eigenvalue as a function of the orbital inclination I. For $I > 90°$, $\theta_3(I) = \theta_3(I - 90°)$.

Note that because of the nodal precession, the eigenvectors are time-dependent.

When $|\dot{\Omega}| \ll \nu_m$, the solution is qualitatively obvious: each component

$$\omega_{(i)} = \boldsymbol{\omega} \cdot \mathbf{u}_{(i)} \tag{24}$$

of the angular rotation vector decreases exponentially with the time constant $(\beta_i \nu_m)^{-1}$; for a generic initial value, $\boldsymbol{\omega}(t)$ asymptotically aligns with the eigenvector $\mathbf{u}_{(3)}$ corresponding to the least eigenvalue β_3. As the node slowly precesses, the faster decay along the two other eigenvectors keeps $\boldsymbol{\omega}$ almost aligned with $\mathbf{u}_{(3)}$. In the limit $\dot{\Omega} = 0$, the one-dimensional subspaces $\{\mathbf{v};\ \mathbf{v} = \lambda \mathbf{u}_{(i)}\}$ are constants of the motion; however, only the subspace generated by $\mathbf{u}_{(3)}$, is stable. Figure 1 gives the angle θ_3 it makes with the polar axis $\mathbf{E}$ as a function of the inclination I. The eigenvalue β_3 is plotted in Figure 2. For LAGEOS, with $I = 109.8°$, $\beta_3 = 1.087$, and $\theta_3 = 28.15°$.

In the opposite, extreme case $|\dot{\Omega}| \gg \nu_m$, the vector $\mathbf{n}$ has a fast rotation and the matrix

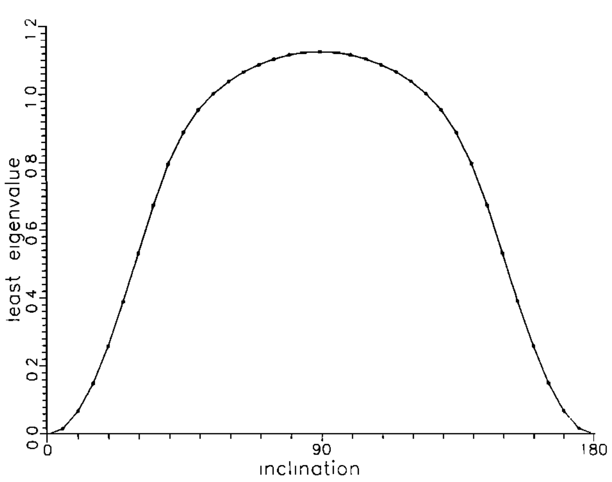

Fig. 2. Least eigenvalue β_3 of the matrix β_{ij} as a function of the orbital inclination.

$$\beta_{ij} = \overline{\beta}_{ij} + \delta\beta_{ij} \tag{25}$$

is the sum of its running average $\overline{\beta}_{ij}$ over a nodal period $P_N = 2\pi/|\dot{\Omega}|$ and a part $\delta\beta_{ij}$ whose average vanishes. Both are of order unity. The same separation can be done for the angular velocity:

$$\boldsymbol{\omega} = \overline{\boldsymbol{\omega}} + \delta\boldsymbol{\omega} \tag{26}$$

and the equation of motion for $\overline{\boldsymbol{\omega}}$ is

$$\frac{1}{\nu_m}\frac{d\overline{\boldsymbol{\omega}}}{dt} = -\overline{\boldsymbol{\beta}} \cdot \overline{\boldsymbol{\omega}} - \overline{\delta\boldsymbol{\beta} \cdot \delta\boldsymbol{\omega}} \tag{27}$$

The order of magnitude of the last term on the right-hand side can be evaluated as follows. $\delta\boldsymbol{\omega}$ satisfies

$$\frac{1}{\nu_m}\frac{d\delta\boldsymbol{\omega}}{dt} + \boldsymbol{\beta}\delta\boldsymbol{\omega} = -\delta\boldsymbol{\beta} \cdot \boldsymbol{\omega} \tag{28}$$

The driving term in the right-hand side prevents the asymptotic decay of $\delta\boldsymbol{\omega}$, but since it almost averages out to zero in a nodal period P_N, only a fraction of the last nodal period will contribute to its value; hence $\delta\omega$ is of order $P_N \nu_m \omega \ll \omega$ and the last term in (27) can be neglected.

The average $\overline{\beta}_{ij}$ is easily obtained from (16) by means of the relationships

$$\overline{n_i} = E_i \cos I \tag{29}$$

$$\overline{n_i n_j} = \tfrac{1}{2} \sin^2 I \delta_{ij} + \frac{3\cos^2 I - 1}{2} E_i E_j \tag{30}$$

$$\overline{\beta}_{ij} = \tfrac{1}{16} [(31 - 42 \cos^2 I + 27 \cos^4 I)\delta_{ij}$$

$$+ (-13 + 78 \cos^2 I - 81 \cos^4 I) E_i E_j] \tag{31}$$

Its eigenvalues are

$$\overline{\beta}_3 = \tfrac{1}{16} (18 + 36 \cos^2 I - 54 \cos^4 I) \tag{32}$$

with its eigenvector oriented along $\mathbf{E}$, and

$$\overline{\beta}_1 = \overline{\beta}_2 = \tfrac{1}{16} (31 - 42 \cos^2 I + 27 \cos^4 I) \tag{33}$$

Now, the explicit expressions (32) and (33) show that $\overline{\beta}_3$ is, for any value of the inclination, the least eigenvalue; hence the spin axis asymptotically aligns with the Earth axis.

The generic case $|\dot{\Omega}| \approx \nu_m$ (relevant for LAGEOS) is better discussed in the reference frame rotating with the node. In this frame the matrix β'_{ij} is equal to the matrix β_{ij} for a fixed node. Hence the rotating components ω'_i of the angular velocity fulfil

$$\frac{1}{\nu_m}\frac{d\omega'_i}{dt} = \frac{\dot{\Omega}}{\nu_m} (\mathbf{E} \times \boldsymbol{\omega}')_i - \beta_{ij}\omega'_j = \frac{\dot{\Omega}}{\nu_m} C_{ij}\omega'_j - \beta_{ij}\omega'_j \tag{34}$$

where

$$C_{ih} = -C_{hi} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{35}$$

is a skew matrix. The eigenvalues of the matrix

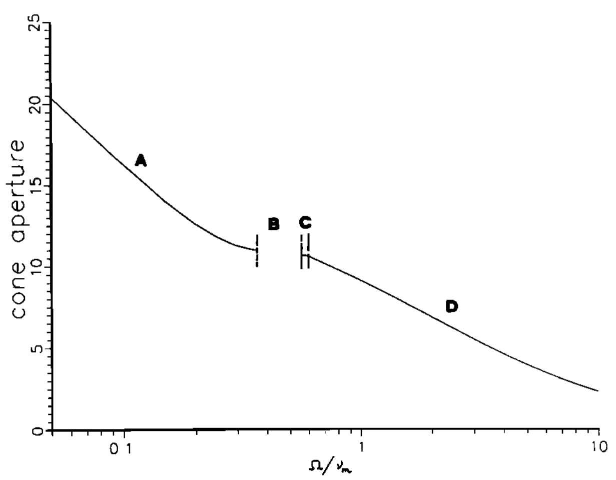

Fig. 3. Aperture of the stable precession cone as a function of the ratio ν_m/Ω for an orbital inclination of 110° or 70°. In regions A and C, all eigenvalues of γ_{ij} are real. In regions B and D, two eigenvalues are complex conjugate; in B their real part is smaller than the third (real) eigenvalue, and hence there is no stable precession cone. In region D the inequality is reversed.

$$\gamma_{ij} = -\beta_{ij} - \frac{\dot{\Omega}}{\nu_m} C_{ij} \tag{36}$$

determine the asymptotic direction of the spin axis. When the eigenvalues are all real, the spin again aligns along the eigenvector corresponding to the least eigenvalue, as in the case $\dot{\Omega} = 0$. When two eigenvalues are complex conjugate, the dynamics is more complicated. The general solution of (34) is

$$\begin{aligned}\boldsymbol{\omega}(t) &= e^{-\beta t}(A\mathbf{u}_1 e^{i\gamma t} + A^*\mathbf{u}_1^* e^{-i\gamma t}) + B\mathbf{u}_3 e^{-\alpha t} \\ &= 2e^{-\beta t}\Re(A\mathbf{u}_1 e^{i\gamma t}) + B\mathbf{u}_3 e^{-\alpha t}\end{aligned} \tag{37}$$

Here α is the real eigenvalue of γ_{ij}, while β, γ are the real and imaginary parts of the complex eigenvalues. The constants $A = \Re(A) + i\Im(A)$ and B are determined by the initial conditions. Equation (36) shows that again, the asymptotic behavior of $\boldsymbol{\omega}$ is determined by the eigenvalue having the smallest real part. If this is the real eigenvalue, the spin axis aligns with the corresponding real eigenvector $\mathbf{u}_3$. In the inertial frame, $\boldsymbol{\omega}$ asymptotically precesses with the period of the node around the Earth's axis. The aperture θ_3' of the precession cone depends on the inclination and the ratio $\dot{\Omega}/\nu_m$. In the opposite case $\beta < \alpha$, for large t,

$$\begin{aligned}\boldsymbol{\omega}(t) &= |A|e^{-\beta t}[\cos(\gamma t + \varphi)\Re(\mathbf{u}_1) - \sin(\gamma t + \varphi)\Im(\mathbf{u}_1)] \\ \varphi &= \arctan(\Im(A)/\Re(A))\end{aligned} \tag{38}$$

and $\boldsymbol{\omega}$ has no asymptotic direction.

This discussion is summarized in Figure 3, where the angle θ_3' is plotted as a function of the parameter $|\dot{\Omega}|/\nu_m$ for an orbital inclination of 110° or 70° (relevant for LAGEOS). In the region A the characteristic equation of the matrix γ_{ij} has three real roots. For small values of $\dot{\Omega}/\nu_m$ the cone aperture tends to the expected asymptotic value of 28.15°. In region B, two eigenvalues are complex conjugate and $\beta < \alpha$, so that there is no asymptotic precession cone. In region C, all eigenvalues become again real. At last, in region D, only one eigenvalue is real and $\alpha < \beta$. For large values of $|\dot{\Omega}|/\nu_m$ the cone aperture tends to zero, as previously shown. In section 3 we provide an estimate of the quantity ν_m for the satellite LAGEOS on the basis of the measures of the spin decay; it turns out that $|\dot{\Omega}|/\nu_m = 1.64$.

4. Gravitational Precession

The oblateness of the spacecraft, determined by the combination

$$\Delta = \frac{C - A}{C} \tag{39}$$

of the principal moments of inertia, produces a precession of the spin vector around the normal $\mathbf{n}$ to the orbital plane with the precession frequency

$$\omega_p = \tfrac{3}{2}\Delta \frac{n^2}{\omega} \cos\varepsilon \tag{40}$$

where n is the mean motion and ε is the "obliquity" of the spacecraft, defined by

$$\cos\varepsilon = \mathbf{n} \cdot \frac{\boldsymbol{\omega}}{\omega} \tag{41}$$

This is the same process that determines the lunisolar, Hipparcos precession of the Earth. We see that since $\omega_p \propto 1/\omega$, the gravitational precession in the end dominates and may make the rotational dynamics chaotic. The mean motion of LAGEOS is $4.65 \times 10^{-4}\ \mathrm{s}^{-1}$, so that the instantaneous precessional period is

$$109.2 \text{ years} \left(\frac{1\ \mathrm{s}}{T}\right)\left(\frac{3.35\%}{\Delta}\right)\left(\frac{1}{|\cos\varepsilon|}\right)$$

where $T = 2\pi/\omega$ is the satellite spin period and $\Delta = 3.35\%$ is the value of the oblateness of LAGEOS quoted in the literature [*Johnson et al.*, 1976]. For a generic obliquity, one would get an initial precessional period (i.e., immediately after launch) of approximately 230 years ($T = 0.6$ s, $\Delta = 1/25$, $\varepsilon = 45°$). The obliquity of the spacecraft was however very close to 90° ($\cos\varepsilon = 0.03$ [*Rubincam*, 1987]), so that the initial precessional period was actually much longer. The magnetic torque has changed the initial obliquity and greatly increased the spin period. Using the spin period and satellite obliquity predicted by the model, the gravitational precessional period was about 5 years in 1989 and is quickly decreasing. For the extreme cases we can again resort to the multiple scale method. When $|\omega_p| \gg (|\dot{\Omega}|, \nu_m)$, the component

$$\boldsymbol{\omega}_\perp = \boldsymbol{\omega} - \omega_\parallel \mathbf{n} \tag{42}$$

of the angular velocity in the orbital plane rotates with an exponentially increasing precession frequency ω_p and has an almost vanishing average; the component $\omega_\parallel = \boldsymbol{\omega} \cdot \mathbf{n}$ along $\mathbf{n}$ is not affected by the gravitational precession and fulfills (equation (20)):

$$\frac{1}{\nu_m}\frac{d\omega_\parallel}{dt} = -(\mathbf{n} \cdot \boldsymbol{\beta} \cdot \mathbf{n})\omega_\parallel = -\beta_\parallel \omega_\parallel = -\tfrac{5}{2}\sin^2 I\,\omega_\parallel \tag{43}$$

TABLE 1. Least Squares Fit of the Data

	$\beta''(0)$	$\beta'(0)$	$(d\beta''/dx)_{x=0}$	T_0	Γ''	Γ'	σ_T/T
A	0.213	···	···	0.55	0.34	···	1.09×10^{-2}
B	0.213	7×10^{-3}	$<10^{-4}$	0.55	0.34	1.5×10^{-2}	1.09×10^{-2}
C	0.220	···	···	0.61	0.26	3.2×10^{-2}	1.61×10^{-2}

Torques are expressed in dyne centimeters.

The angular speed ω is only slightly affected by ω_p and fulfils

$$\frac{1}{\nu_m}\,\omega\,\frac{d\omega}{dt} = -\boldsymbol{\omega}\cdot\boldsymbol{\beta}\cdot\boldsymbol{\omega} \tag{44}$$

What counts is its average over a precession period, obtained from the obvious relationship (valid up to terms of order ν_m/ω_p)

$$(\boldsymbol{\omega\omega})_p = \omega_\parallel^2\mathbf{nn} + \langle\boldsymbol{\omega}_\perp\boldsymbol{\omega}_\perp\rangle = \tfrac{1}{2}\omega^2[\sin^2\varepsilon\mathbf{1} - (3\cos^2\varepsilon - 1)\mathbf{nn}] \tag{45}$$

and (20):

$$\frac{1}{\nu_m}\frac{d\omega}{dt} = -\tfrac{1}{4}\,\omega[5 - \cos^2 I + \cos^2\varepsilon(5 - 9\cos^2 I)] \tag{46}$$

It can be confirmed that the right hand side is positive, except for $\cos^2\varepsilon = \cos^2 I = 1$, for which it vanishes.

From (46) and (43) the obliquity ε fulfils

$$\frac{1}{\nu_m}\frac{d\cos\varepsilon}{dt} = \tfrac{1}{4}\cos\varepsilon\,\sin^2\varepsilon(9\cos^2 I - 5) \tag{47}$$

When $I < I_0 = \arccos\,(5/9)^{1/2} = 41.8°$, the spin axis asymptotically aligns with the normal to orbital plane ($\varepsilon \to 0$); when $I > I_0$, the spin axis ends up rotating in the orbital plane at an increasing rate.

5. THE ANGULAR VELOCITY OF LAGEOS

It is interesting to apply the previous theory to the time evolution of the spin axis of the satellite LAGEOS. In the inertial frame, the equation of motion to be integrated are

$$\frac{1}{\nu_m}\frac{d\omega_i}{dt} = -\beta_{ij}(t)\omega_j + \frac{\omega_p}{\nu_m}\,\varepsilon_{ijk}\omega_j n_k \tag{48}$$

Here we have introduced the gravitational torque $\omega_p\boldsymbol{\omega}\times\mathbf{n}$, which is nonlinear in the components of the angular velocity.

There are at least two methods for measuring the spin vector of LAGEOS. With infrared radar, one can use the return of four out of the 426 retroreflectors which are made of Germanium and suitable for operations at 10.6 μm. By analyzing the Doppler return frequency as a function of time, *Sullivan* [1980] measured a period of 1.44 s on April 11, 1979. A military microwave Doppler radar was used by E. M. Gaposchkin (private communication of May 14, 1987, to R. Kolenkiewicz) of Lincoln Laboratory, Massachusetts Institute of Technology, who provided a plot with several measurements of the spin period from 1977 to 1983. The measurements are rather dense from March 1982 to November 1983 and show a big gap after April 1979 until resumption in 1982. A single measurement is available on March 20, 1987. We thank E. M. Gaposchkin for also giving us a copy of this plot, but regret that because of military classification rules, it was not accompanied by numerical results and their formal errors. Also, we have no information about the spin direction. As elementary considerations show, because of the modulation of the Doppler return line at the rotation frequency, a microwave radar measurement of the spin of LAGEOS can be made fairly easily with a reasonable antenna; it is difficult to recognize a military interest in such a measurement. A similar method was used, for example, to determine the rotation of Venus with the Arecibo, Haystack, and Goldstone radar systems [*Shapiro et al.*, 1979; *Shapiro*, 1967], for which the characteristic ratio surface/distance[4] was much smaller. We hope that these measurements will be periodically repeated for LAGEOS in the future.

We have numerically integrated (48) and fitted the available measurements of the spin period with the parameter $\beta''(0)$ (see (12) and (21)) and the initial spin period T_0 (the value $T_0 = 0.6$ s given by *Ordhal* [1975] was nominal). The fit formally used the variable log T. We obtained

$$\beta''(0) = 0.213 \qquad T_0 = 0.55\text{ s} \tag{49}$$

The estimate of T_0 is in good agreement with the nominal value and compatible with the performance of the third stage of the Delta launcher used for LAGEOS. The value for $\beta''(0)$ can be translated into an effective value for the conductivity σ; indeed, with this low-frequency approximation, the magnetic torque is equivalent to the magnetic torque experienced by a uniform sphere with radius ρ and effective conductivity σ_{eff} such that (see (12))

$$\sigma_{\text{eff}} = \sigma\beta''(0) = 4.8\;10^{16}\text{ s}^{-1} \tag{50}$$

Hence

$$\delta_{\text{eff}}/\rho = \delta/\rho[\beta''(0)]^{1/2} = 0.57(T/0.6)^{1/2} \tag{51}$$

which supports our "low-frequency" approximation. For a further check, besides the low-frequency limit discussed so far (model A), we have fitted the available data with two more models: Model B uses first-order corrections (i.e., to $O(\delta^2/\rho^2)$ to the low-frequency limit, using the full expression (3′) for the torque and also estimating $\beta'(0)$ and $(d\beta''/dx)_{x=0}$ besides $\beta''(0)$ and T_0, and model C is the "equivalent sphere". We use full Landau-Lifshitz expression for the polarizability α and fit the data with T_0 and the parameter $\beta''(0)$ defined by (50).

In Table 1 we summarize the results of the fit, giving, besides the values of the fitted parameters, the magnitude (in dyne centimeters) of the two components of the magnetic torque (see equation (3′)) at $t = 0$ and the residual of the spin period

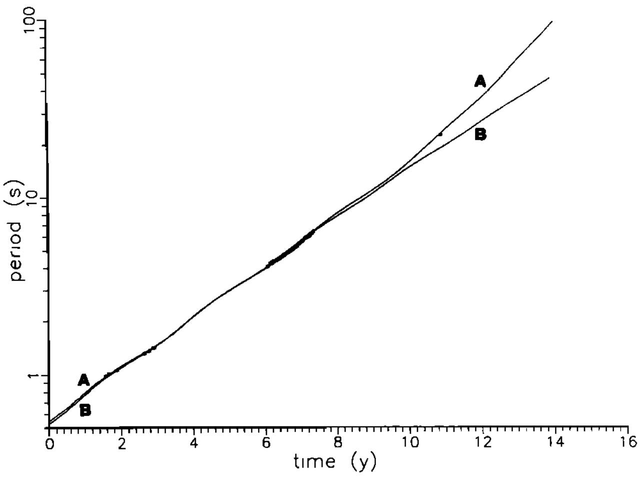

Fig. 4. Predicted evolution of the LAGEOS spin period as a function of time (years), under the action of magnetic and gravitational torques. The dots represent the measured values of the period. Curve A refers to a southward orientation of $\boldsymbol{\omega}(0)$, while curve B, which shows a poorer agreement, refers to the opposite orientation.

$$\frac{\sigma_T}{T} = \frac{\left[\sum_i (\ln T_i - \ln T(t_i))^2\right]^{1/2}}{N^{1/2}} \tag{52}$$

Figure 4 shows, besides the measured values, the predicted spin period increase obtained from the numerical integration of (48), using the initial orientation of the spin axis

$$\theta = 158° \qquad \phi = 104°$$

in a frame of reference with the x axis oriented along the line of the nodes at $t = 0$ and z axis along the Earth's axis. The above values are obtained from *Rubincam* [1987]. The very accurate fit is an excellent confirmation of the appropriateness of the model.

Under the reversal of the sign of $\boldsymbol{\omega}$, the magnetic torque (12) is odd and the gravitational torque is unchanged. Hence the initial conditions $\boldsymbol{\omega}(0)$ and $-\boldsymbol{\omega}(0)$ evolve differently if the gravitational torque is important. Indeed, it turns out that a southward orientation of the initial spin vector gives rise to a faster decay of the spin period. A best fit of the experimental data with a northward orientation of the initial spin vector would lead to a much poorer agreement (see Figure 4). *Rubincam* [1987] leaves the ambiguity in the orientation of the initial spin vector unresolved. The present model gives a precise indication about the sense of rotation, which, if confirmed, will increase our confidence in its appropriateness.

The predicted evolution of the spin vector obtained from (48) can be read off from Figure 5*a*, where the projection of $\boldsymbol{\omega}/\omega$ on the equatorial plane is plotted starting from the initial position at $t = 0$ and propagated for 14 years. For comparison, we have reported also the curve obtained from model C (dashed line). The very similar behavior of the spin vector in both cases indicates, again, that the total torque can be safely assumed to be linear in ω (i.e., $\Gamma' = 0$). Two successive marks on the curve are separated by 1 year. Figure 5*b* is obtained when the gravitational precession is turned off (this is the case of a satellite with $\Delta = 0$). $\boldsymbol{\omega}$ now has a spiralling periodic component due to the node precession and drifts toward the origin. For large t, $\boldsymbol{\omega}$ reaches a stable precession cone of aperture 7.5°, as discussed in section 3 (see also Figure 3). Comparison between Figures 5*a* and 5*b* shows the relevance of the gravitational precession for orientation of the spin axis. Figure 5*a* shows also the

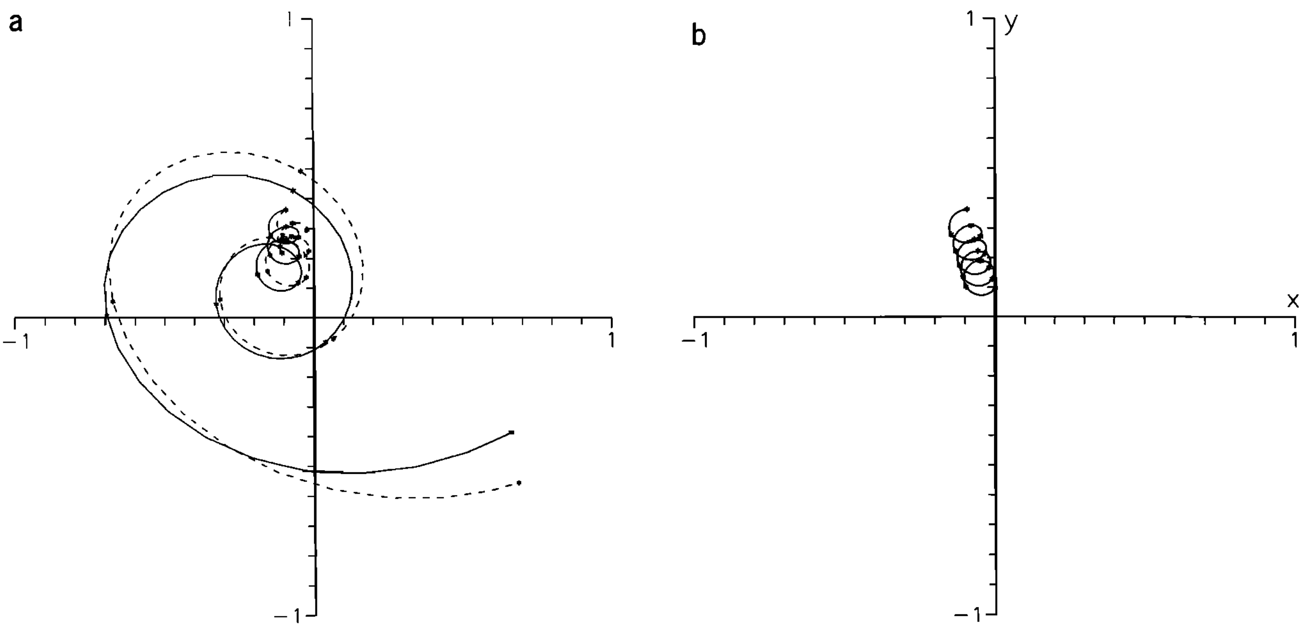

Fig. 5. Projection of the (unitary) spin vector of LAGEOS on the equatorial plane at successive times for 14 years since launch. (*a*) Magnetic and gravitational torques (the dashed curve is obtained from model C (equivalent sphere)); (*b*) magnetic torque only. Marks are spaced by 1 year. The x axis is oriented along the nodal line at $t = 0$.

difficulty of reliable propagations of the spin vector for long integration times. This is a consequence of the peculiar coupled actions of the magnetic and gravitational torques, which later make the satellite dynamics very sensitive to the initial conditions.

All models give similar results and show a good agreement with the observations. In particular, both model B and model C show that the torque Γ'' largely dominates over Γ', thus confirming our initial guess. For model C, the results of the fit give an effective conductivity of $\sigma_l = 4.9 \times 10^{16}$ s. (i.e., more than 4 times less than the bare conductivity of the outer shell) and a penetration depth $\delta_l = 16.9$ cm, for $T = 0$, a value 2.13 times larger than the one quoted in equation (2).

It is interesting to compare the quantities T_0 and Γ'' obtained in the three cases. Using models A and B, one obtains a larger torque in the initial phase of the decay. This excess of dissipation "forces" the parameter T_0 to a lower value. Model C has initially a good agreement with the observations and a spin period equal to the nominal one (0.6 s), but in the end the total variance is larger than for the other models. This indicates that higher-order terms of the magnetic polarizability may come into play in the early phase of the slowing down. However, the above results provide a convincing justification to the use of the low-frequency limit. It is also remarkable that within a couple of years from launch all models indicate essentially the same period and total torque. The large discrepancy in the value of Γ' between models A and B can be explained with the fact that the torque Γ', being orthogonal to $\boldsymbol{\omega}$, does not affect directly the spin period and therefore the determination of $\beta'(0)$ is more difficult than the one of $\beta''(0)$. It must be mentioned, however, that the least square function of model B exhibits a well-marked minimum for the values of Table 1. Note also that using the same model B, the value of $(d\beta''/dx)_{x=0}$ cannot be reliably determined and is smaller, in absolute value, than 10^{-4}. This fact may be related to a general property of the magnetic polarizability of a homogeneous sphere, whose expansion in series of the parameter $(\rho/\delta)^2$ contains only terms of order $2n$ and $2n + 1$ for the real and imaginary parts (α' and α''), respectively. Hence for a uniform sphere, $(d\beta''/dx)_{x=0} = 0$. Such a result may be valid, more generally, for symmetric conductors spinning around the axis of symmetry.

6. CONCLUSIONS

In this paper we have developed a theoretical model for the evolution of the angular velocity of an axially symmetric, conducting satellite orbiting in a dipole magnetic field. Although the treatment is quite general and some results are interesting in themselves, the present work was driven by the need to model the very tiny forces acting on the laser-tracked satellite LAGEOS. As discussed in the introduction, some of those forces depend indeed on the orientation of the spin axis of the satellite.

The predictions of the model are in excellent agreement with the available measurements of the slowing down of LAGEOS. Although the spin period shows a roughly exponential decay that can be predicted on the basis of simple dimensional considerations, some features of the experimental curve are related to more complex phenomena, due to the peculiar combined action of magnetic and gravitational torque. The confidence in the model stems from its capability to also reproduce very well those features not simply related to a bare exponential trend. Moreover, the model is sensitive to the sign of the initial angular velocity: If the wrong sign is used in the initial conditions, the agreement with the data is much poorer.

It would be interesting to use the evolution of the angular velocity, as predicted by our theory, for an improved modelling of the so-called "radiation rocket" effect. In view of this application, we think, however, that measurements of the vectorial angular velocity would be very important for a more complete and exhaustive test of the model. Recently [*Rubincam*, 1990], an attempt to recover the position of the spin axis of LAGEOS has been made by fitting anomalous along-track acceleration data with a model of the Yarkovsky thermal drag. Rubincam's conclusion is that the spin axis has slightly drifted from its initial position toward the axis of the Earth, in the period 1976–1987. This is roughly in agreement with our results (see Figure 5*a*).

The evolution of the angular velocity, which up to now has been determined essentially by the magnetic torque, will exhibit in the future a much more complex and unpredictable behavior, as the gravitational precession due to the slight oblateness of the satellite will in the end dominate its dynamics. The exponential slowing down due to eddy currents will continue with approximately the same rate, and the period of the gravitational precession, being proportional to ω, will become very short. Our model predicts that the transition to the new regime, dominated by the gravitational torque, will occur within 1–2 years from now (December 1990) and LAGEOS, like a very slow top, will start tumbling faster and faster, with a chaotic dynamics. When the time scale of variation of $\boldsymbol{\omega}$ will become comparable with the orbital period of the satellite, the theory developed in this paper will be no longer applicable.

APPENDIX A

Due to the magnetic declination, β_{ij} has a daily variation which can be averaged out; the orbital and the day averages can be separately performed if the corresponding frequencies are separated and do not show small integer commensurability. The second average is again done by symmetry. Let D be the angle between the Earth axis $\mathbf{E}$ and the magnetic dipole $\mathbf{d}$. Decomposing $\mathbf{d}$ in its polar and equatorial components

$$\mathbf{d} = \mathbf{E}d \cos D + \mathbf{d}' \quad \text{(A1)}$$

we have $\langle \mathbf{d}' \rangle = \mathbf{0}$ and

$$\langle \mathbf{d}'\mathbf{d}' \rangle_{\text{day}} = \tfrac{1}{2} d^2 \sin^2 D(\mathbf{1} - \mathbf{EE}) \quad \text{(A2)}$$

$$\langle \mathbf{dd} \rangle_{\text{day}} = d^2 \cos^2 D\mathbf{EE} + \langle \mathbf{d}'\mathbf{d}' \rangle_{\text{day}}$$
$$= \tfrac{1}{2} d^2[\sin^2 D\mathbf{1} + (3 \cos^2 - 1)\mathbf{EE}] \quad \text{(A3)}$$

The day average of β_{ij} will contain the true inclination I, given by equation (7) so that

$$\langle \cos^2 I_M \rangle_{\text{day}} = \tfrac{1}{2} (1 - 3 \cos^2 D) \cos^2 I + \tfrac{1}{2} \sin^2 D$$

We obtain

$$\langle \beta_{ij} \rangle_{\text{day}} = \tfrac{1}{8} [1 - \tfrac{5}{2} \sin^2 D - 3(1 - \tfrac{3}{2} \sin^2 D) \cos^2 I]\delta_{ij}$$

$$-\frac{1}{8}(2-3\sin^2 D)E_iE_j$$

$$+\frac{9}{8}[(1-\frac{3}{2}\sin^2 D)\cos^2 I-\frac{3}{2}\sin^2 D]n_in_j \quad \text{(A4)}$$

For a precise work one could use this expression instead of equation (16)

APPENDIX B

We wish to evaluate the average of the term

$$B_iB_j=\frac{(3r_ir_kd_k-r^2d_i)(3r_jr_hd_h-r^2d_j)}{r^{10}} \quad \text{(B1)}$$

over a circular orbit of radius r in a plane with unit normal $\mathbf{n}$. Symmetry demands that

$$\langle r_ir_j\rangle=\frac{1}{2}r^2(\delta_{ij}-n_in_j) \quad \text{(B2)}$$

The average of $r_ir_jr_kr_h$, a fully symmetric tensor constructed only with Kronecker's tensor δ_{ij} and the vector n_i, must have the general expression

$$\langle r_ir_jr_kr_h\rangle=\frac{1}{3}r^4\alpha(\delta_{ij}\delta_{kh}+\delta_{ih}\delta_{jk}+\delta_{ik}\delta_{jh})$$

$$+\frac{1}{6}r^4\beta(\delta_{ij}n_kn_h+\delta_{ih}n_kn_j+\delta_{ik}n_hn_j+\delta_{kh}n_in_j$$

$$+\delta_{jk}n_in_h+\delta_{hj}n_in_k)+r^4\gamma n_in_jn_kn_h \quad \text{(B3)}$$

The dimensionless coefficients α, β, and γ are determined by the requirements (not independent)

$$\langle r_ir_jr_kr_h\rangle=r^2\langle r_ir_j\rangle=\frac{1}{2}r^4(\delta_{ij}-n_in_j)$$

$$\langle r_ir_jr_kr_h\rangle n_h=0$$

They give

$$\alpha=\frac{3}{8} \qquad \beta=-\frac{3}{4} \qquad \gamma=\frac{3}{8} \quad \text{(B4)}$$

and

$$\langle r_ir_jr_kr_h\rangle=\frac{1}{3}r^4\,\frac{3}{8}(\delta_{ij}\delta_{kh}+\delta_{ih}\delta_{jk}+\delta_{ik}\delta_{jh})$$

$$-\frac{1}{6}r^4\,\frac{3}{4}(\delta_{ij}n_kn_h+\delta_{ih}n_kn_j+\delta_{ik}n_hn_j+\delta_{kh}n_in_j$$

$$+\delta_{jk}n_in_h+\delta_{hj}n_in_k)+r^4\,\frac{3}{8}\,n_in_jn_kn_h \quad \text{(B5)}$$

After some algebra

$$r^3\langle B_iB_j\rangle=\frac{9}{8}d^2\sin^2 I_M\delta_{ij}+\frac{1}{4}d_id_j+\frac{1}{8}(27\cos^2 I_M-9)\,d^2n_in_j$$

$$-\frac{3}{4}(d_in_j+d_jn_i)\cos I \quad \text{(B6)}$$

where

$$\cos I_M=\frac{\mathbf{d}\cdot\mathbf{n}}{d} \quad \text{(B7)}$$

gives the "magnetic" inclination of the orbit. Taking the trace

$$r^6\langle B^2\rangle=(\frac{20}{8}-\frac{3}{2}\cos^2 I_M)\,d^2 \quad \text{(B8)}$$

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