Electromagnetic waves in vacuum.

The discovery of displacement currents entails a peculiar class of solutions of Maxwell equations: travelling waves of electric and magnetic fields in vacuum. In the absence of currents and charges, the equations governing electric and magnetic field are:

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

Taking the curl of (3) and using (4):

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Using (1) we get D'Alambert equation: $\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$

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Each of the two equation pairs lead to the D'Alambert eq. (respectively for E_x and B_y, and E_y and B_x). For example, the first pair combines into:

$$\frac{\partial^2 E_x}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} = 0 \qquad \frac{\partial^2 B_y}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 B_y}{\partial t^2} = 0 \qquad (\mathbf{E} \perp \mathbf{B}\,!)$$

$$\Rightarrow \quad E_x = E_x(z - ct) \qquad B_y = B_y(z - ct)$$

The linearity of D'Alambert eq. entails that the generic EM plane wave can be seen as the superposition of sinusoidal waves. Let us the assume that each component has the form

$$u = u_0 \exp[i(kz - \omega t + \phi_u)]$$

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The pair of equations for (E_x,B_y) yields

$$kE_{0x}\exp[i(kz-\omega t+\phi_E)]=\frac{\omega}{c}B_{0y}\exp[i(kz-\omega t+\phi_B)]$$

$$kB_{0y}\exp[i(kz-\omega t+\phi_B)]=\frac{\omega}{c}E_{0x}\exp[i(kz-\omega t+\phi_E)]$$

These equations must be verified for any t and z. Therefore

$$E_{0x}=B_{0y} \qquad \phi_E=\phi_B$$

The same conditions are true for (E_y,B_x). At each (t,z) the magnitude of the electric and magnetic field is the same (in CGS-units) .

Polarization of EM waves

The two classes of solutions (E_x,B_y) and (E_y,B_x) are independent: they represent the two polarization modes of EM radiation. As the **E** and **B** fields lie on a plane, these modes correspond to linear polarizations.
A generic (unpolarized) EM wave is a superposition of the two modes, with different phases and amplitudes. For example, introducing the unit vectors along x and y:

$$\mathbf{E}(z,t) = E_{0x}\cos(kz-\omega t+\phi_x)]\hat{\mathbf{x}} + E_{0y}\cos(kz-\omega t+\phi_y)]\hat{\mathbf{y}}$$

Instead of the linearly polarized modes, one could use two circularly polarized modes:

$$E_x = E_y \qquad \phi_y = \phi_x + \frac{\pi}{2}$$

Left Circular Polarization (LCP)

and

$$E_x = E_y \qquad \phi_y = \phi_x - \frac{\pi}{2}$$

Right Circular Polarization (RCP)

Maxwell Equations in a Plasma

The wave propagation is governed by Maxwell equations (CGS-Gauss):

$$\nabla \cdot \mathbf{E} = 4\pi\rho \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

We will solve the combined set of Maxwell equations and electron equation of motion using 1) a <u>perturbative approach</u> (to first order) and 2) <u>normal modes</u>.

1) All quantities are the sum of an unperturbed (background) value and a small perturbation:

$n = n_0 + n_1$ $\qquad$ $\mathbf{v} = \mathbf{v}_1$ (quiescent plasma)

$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$ $\qquad$ $\mathbf{E} = \mathbf{E}_1$ (no external electric field)

Maxwell Equations in a Plasma.

2) All perturbations are periodic and depend on space and time as

$$u = u_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

The surfaces of constant phase move in space at the phase velocity

$$v_\phi = \frac{\omega}{k}$$

(suggestion: consider the surfaces with null phase and compute r/t)

The set of differential equations becomes now a set of algebraic equations; the differential operators transform as

$$\nabla \rightarrow i\mathbf{k} \qquad \nabla \cdot \rightarrow i\mathbf{k} \cdot \qquad \nabla \times \rightarrow i\mathbf{k} \times \qquad \frac{\partial}{\partial t} \rightarrow -i\omega$$

The equations of motion.

Navier-Stokes equation for the electrons (collisionless plasma):

$$m_e n_e \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla p - e n_e \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right)$$

The space charge density $\rho = \text{-}en_e$ and current density $\mathbf{J}$ = -$en_e\mathbf{v}$ fullfil the equation of continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

We assume that the plasma the plasma is magnetized: a constant and homogeneous magnetic field is superimposed to the magnetic field of the wave. The plasma is supposed to be isothermal, so we do not need to consider the energy equation. (Not only: we will see soon that the plasma may be safely supposed to be cold, with T = 0**!**)

The linearized, normal mode equations.

The Maxwell equations, charge conservation and electron equation of motion become:

$$i\mathbf{k}\cdot\mathbf{E}_1 = 4\pi e n_1 \qquad (1)$$

$$\mathbf{k}\cdot\mathbf{B}_1 = 0 \qquad (2)$$

$$\mathbf{k}\times\mathbf{E}_1 = \frac{\omega}{c}\mathbf{B}_1 \qquad (3)$$

$$\mathbf{k}\times\mathbf{B}_1 = -i\frac{4\pi}{c}\mathbf{J} - \frac{\omega}{c}\mathbf{E}_1 \qquad (4)$$

$$e\omega n_1 = \mathbf{k}\cdot\mathbf{J} \qquad (5)$$

$$-im_e n_0\omega\,\mathbf{v}_1 = -en_0\left(\mathbf{E}_1 + \frac{1}{c}\mathbf{v}_1\times\mathbf{B}_0\right) \qquad (6)$$

Note that now the first two Maxwell equations (1-2) follow from the last pair (3-4) and charge conservation (5): (2) follows by taking the scalar product of (3) by **k**; (1) follows by taking the scalar product of (4) by **k** and using (5).

The conductivity tensor.

Eq. (6) entails a linear relation between current density and electric field.

$$i\omega \mathbf{J} = -\frac{e^2 n_0}{m_e}\mathbf{E}_1 + \frac{e}{m_e c}\mathbf{J}\times\mathbf{B}_0$$

or, writing $\mathbf{B_0}$=$B_0\mathbf{n}$ and introducing the plasma and cyclotron frequencies,

$$\mathbf{J} = i\frac{\omega_p^2}{4\pi\omega}\mathbf{E}_1 - i\frac{\Omega_c}{\omega}\mathbf{J}\times\mathbf{n}$$

Expanding the vector product, one gets the generalized Ohm’s law and the conductivity tensor $\boldsymbol{\sigma}$:

$$\mathbf{J} = \boldsymbol{\sigma}\cdot\mathbf{E}$$

The conductivity tensor.

$$\boldsymbol{\sigma} = i\frac{n_e e^2}{\omega m\left(1-\left(\frac{\Omega_c}{\omega}\right)^2\right)}\begin{bmatrix} 1 & -i\frac{\Omega_c}{\omega} & 0 \\ i\frac{\Omega_c}{\omega} & 1 & 0 \\ 0 & 0 & \left(1-\left(\frac{\Omega_c}{\omega}\right)^2\right) \end{bmatrix}$$

The conductivity tensor is therefore anti-hermitean $\sigma_{ij} = -\sigma^*_{ji}$. Therefore no heat is dissipated, as the real part of $\boldsymbol{\sigma}$ is skew and $\left\langle \mathrm{Re}(E_i)\,\mathrm{Im}(E_j)\right\rangle = 0$

$$Q = \mathrm{Re}(\mathbf{E})\cdot\mathrm{Re}(\mathbf{J}) = 0 \qquad \mathrm{Re}(J_i) = \mathrm{Re}(\sigma_{ij})\,\mathrm{Re}(E_j) - \mathrm{Im}(\sigma_{ij})\,\mathrm{Im}(E_j)$$

Plane EM waves in a plasma.

The refractive index of a medium is defined as the ratio between *c* and the phase velocity

$$n_r = \frac{c}{v_\phi} = \frac{ck}{\omega}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

By combining Maxwell eq.

$$\mathbf{k}\times\mathbf{E}_1 = \frac{\omega}{c}\mathbf{B}_1 \qquad \mathbf{k}\times\mathbf{B}_1 = -i\frac{4\pi}{c}\mathbf{J} - \frac{\omega}{c}\mathbf{E}_1$$

one gets:

$$(1-n_r^{\,2})E_i + n_r^2\frac{k_i k_j}{k^2}E_j = -i\frac{4\pi}{\omega}J_i \qquad (7)$$

which gives the current as a function of the electric field. Making use of the relationship $\mathbf{J} = \boldsymbol{\sigma}\cdot\mathbf{E}$ one obtains

$$(1-n_r^{\,2})E_i + n_r^2\frac{k_i k_j}{k^2}E_j = i\frac{4\pi}{\omega}\sigma_{ij}E_j \;\longrightarrow\; \left((1-n_r^{\,2})\delta_{ij} + n_r^2\frac{k_i k_j}{k^2} - i\frac{4\pi}{\omega}\sigma_{ij}\right)E_j = 0$$

Plane EM waves in a plasma.

The homogeneous set of equations has non-trivial solutions iff

$$\det\left((1-n_r^{\;2})\delta_{ij} + n_r^2 \frac{k_i k_j}{k^2} - i\frac{4\pi}{\omega}\sigma_{ij} \right) = 0$$

This is the dispersion equation of the plasma.

EM waves in a plasma: **B**=0

If **B**=0, the conductivity tensor reduces to a scalar:

$$\sigma = -i\frac{\omega_p^2}{4\pi\omega}$$

Maxwell eq. allow two modes of propagation:

(1) $\mathbf{k} \parallel \mathbf{E}_1$ and (2) $\mathbf{k} \perp \mathbf{E}_1$

(1) $i\omega\mathbf{E} + 4\pi\,\mathbf{J} = i\omega\mathbf{E} + 4\pi\,\sigma\mathbf{E} = i(\omega - \omega_p^2/\omega)\,\mathbf{E} = 0 \quad \Rightarrow \quad \omega^2 = \omega_p^2$

Longitudinal plasma waves occur at the well known plasma frequency.
The wavenumber is free!

(2) $(1 - n_r^2)\mathbf{E} = i\dfrac{4\pi}{\omega}\mathbf{J}$

EM waves in a plasma: **B**=0

(2) $\mathbf{k} \perp \mathbf{E}_1$

$$(1-n_r^2)\mathbf{E} = i\frac{4\pi}{\omega}\mathbf{J}$$

Substituting $\mathbf{J} = \sigma\mathbf{E} = -i\frac{\omega_p^2}{4\pi\omega}\mathbf{E}$ one gets

$$\left(1-n_r^2-\frac{\omega_p^2}{\omega^2}\right)\mathbf{E} = 0 \quad \Rightarrow \quad n_r = \sqrt{1-\frac{\omega_p^2}{\omega^2}}$$

The index of refraction of a plasma is smaller than unity. This implies that the phase velocity of e.m. waves is larger than *c:*

$$v_\phi = \frac{c}{n_r} = \frac{\omega}{k} > c$$

EM waves in a plasma: **B**=0

However, information does not travel at the phase velocity: a continuous, monochromatic, plane wave (starting at t=-∞) cannot be used to convey information. Information (e.g. electromagnetic pulses) travels at the group velocity (always smaller or equal to c):

$$v_g = \frac{d\omega}{dk} \qquad\qquad v_\phi = \frac{c}{n_r} = \frac{\omega}{k} > c$$

In an unmagnetized plasma:

$$n_r = \frac{ck}{\omega} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \rightarrow \quad \omega^2 = c^2 k^2 - \omega_p^2 \quad \rightarrow \quad v_g = \frac{d\omega}{dk} = c n_r$$

Group velocity

Let us consider a superposition of two EM waves with slightly different frequency and wavenumber:

$$u_1 = \cos[(k+\Delta k)z - (\omega + \Delta\omega)t]$$
$$u_2 = \cos[(k-\Delta k)z - (\omega - \Delta\omega)t]$$

The superposition of the two waves is:

$$\begin{aligned} u_1 + u_2 &= \cos[(k+\Delta k)z - (\omega+\Delta\omega)t] + \cos[(k-\Delta k)z - (\omega-\Delta\omega)t] = \\ &\cos[(kz-\omega t) + (\Delta kz - \Delta\omega t)] + \cos[(kz-\omega t) - (\Delta kz - \Delta\omega t)] = \\ &\cos(kz-\omega t)\cos(\Delta kz - \Delta\omega t)] - \sin(kz-\omega t)\sin(\Delta kz - \Delta\omega t)] + \\ &\cos(kz-\omega t)\cos(\Delta kz - \Delta\omega t)] + \sin(kz-\omega t)\sin(\Delta kz - \Delta\omega t)] = \\ &2\cos(kz-\omega t)\cos(\Delta kz - \Delta\omega t)] \end{aligned}$$

The low frequency modulation travels at the speed $v_g = \dfrac{\Delta\omega}{\Delta k}$

Group velocity

For $\Delta\omega, \Delta k \to 0 \quad v_g = d\omega / dk$

Let us consider now a "wave packet", i.e. a more general superposition of EM waves with different frequency and wavenumber:

$$\Psi(z,t) = \int_{k_0-\Delta k}^{k_0+\Delta k} A(k) \exp i[kz - \omega t]\, dk$$

$$\cong \int_{k_0-\Delta k}^{k_0+\Delta k} \left[A(k_0) + \frac{1}{2}\left(\frac{d^2 A}{dk^2}\right)_{k_0} \xi^2 \right] \exp i[(k_0+\xi)z - \omega(k_0)t - \left(\frac{d\omega}{dk}\right)_{k_0} \xi t]\, d\xi$$

$$\cong A(k_0) \exp i[k_0 z - \omega(k_0)t] \int_{k_0-\Delta k}^{k_0+\Delta k} \exp i[\xi z - \left(\frac{d\omega}{dk}\right)_{k_0} \xi t]\, d\xi$$

$$\cong 2A(k_0) \exp i[k_0 z - \omega(k_0)t] \frac{\sin[z - \left(\frac{d\omega}{dk}\right)_{k_0} t]}{z - \left(\frac{d\omega}{dk}\right)_{k_0} t}$$

Group velocity

This "pulse" travels at the speed $v_g = \frac{d\omega}{dk}$

The superposition of the two waves is: